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MATHEMATICAL STUDY OF THE SMALL OSCILLATIONS OF A CATENOIDAL LIQUID BRIDGE BETWEEN TWO EQUAL MEMBRANES UNDER ZERO GRAVITY

P. Capodanno

Laboratoire de Mathématiques et de Mécanique Théorique, UFR des Sciences et des Techniques F 25030 Besançon Cedex, France

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This paper deals with the mathematical study of the problem of small oscillations of a liquid bridge with a free surface, held together by surface tension between two elastic coaxial circular disks considered as membranes, under zero gravity, in the case of a catenoidal liquid bridge in the equilibrium position. The equations, which give the reciprocal displacements of the free surface and of the membranes, are reduced to a variational equation. The existence of the eigenfrequencies depends on the coerciveness of the bilinear form which appears in this equation. It is shown that these eigenfrequencies exist under two simple conditions involving the ratio between the distance of the planes of the disks and the tensions of the membranes.

1. INTRODUCTION

Since the beginning of research into hydroelasticity under microgravity conditions, the behaviour of liquid bridges between two generally circular coaxial disks has been extensively studied. This interest is due, in particular, to the applications of this geometry to crystal growth from floating zones. A very good survey of this research can be found in the paper of Sanz (1992).

In fabrication processes under microgravity conditions, such as crystal growth, the oscillations of the free liquid surface is often of detrimental effect on the products. These oscillations are excited by the vibrations of the supporting disks. Consequently, it is important to study the effects of the elasticity of the disks.

The mathematical problem of the stability and of the small oscillations of an inviscid incompressible fluid mass in a container under zero gravity has been treated in the books by Moiseyev & Rumiantsev (1968), Myskhis *et al.* (1987) and Kopachevskii *et al.* (1989) by means of the methods of functional analysis.

It is well-known that, in an equilibrium position, the liquid-free surface is a surface with constant mean curvature. It is possible to determine analytically and geometrically these surfaces, if they are surfaces of revolution (Delaunay 1841). In particular, if the mean curvature is zero (minimum surface), we obtain a catenoid.

Using a method which is different from the method of the Russian scientists cited, the author has studied the small oscillations of a catenoidal bridge between two rigid parallel plates in two cases: (i) with the contact angles constant and equal (Capodanno 1994); and (ii) with the edges anchored (Capodanno 1995).

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In this paper, in order to take into account the vibrations of the supporting disks, the mathematical problem of the small oscillations of a catenoidal liquid bridge under zero gravity with a free surface, held together by a constant surface tension between two elastic coaxial circular disks considered as membranes, is studied.

The equations, which give the displacements of the free surface and of the membranes, are reduced to a variational equation. The existence of the eigenfrequencies of the system depends essentially on the coerciveness of the bilinear form which appears in this equation. This problem can be reduced to another eigenvalue problem, and it can be shown that the form is coercive if the smallest eigenvalue is strictly greater than one. After a long and careful discussion, it is possible to prove the existence of the eigenfrequencies of the system if two simple inequalities are satisfied, which express that the ratio between the distance of the centres of the disks and the neck radius of the catenoid must be sufficiently small and the tensions of the membranes sufficiently great.

2. FORMULATION

We consider a liquid bridge between two elastic equal coaxial disks S_1 and S_2, the edges of which are the equal circles C_1 and C_2, considered as membranes, in the case where the free surface of the liquid in the equilibrium position S_0 is a catenoid, the membranes being in the planes of the disks.

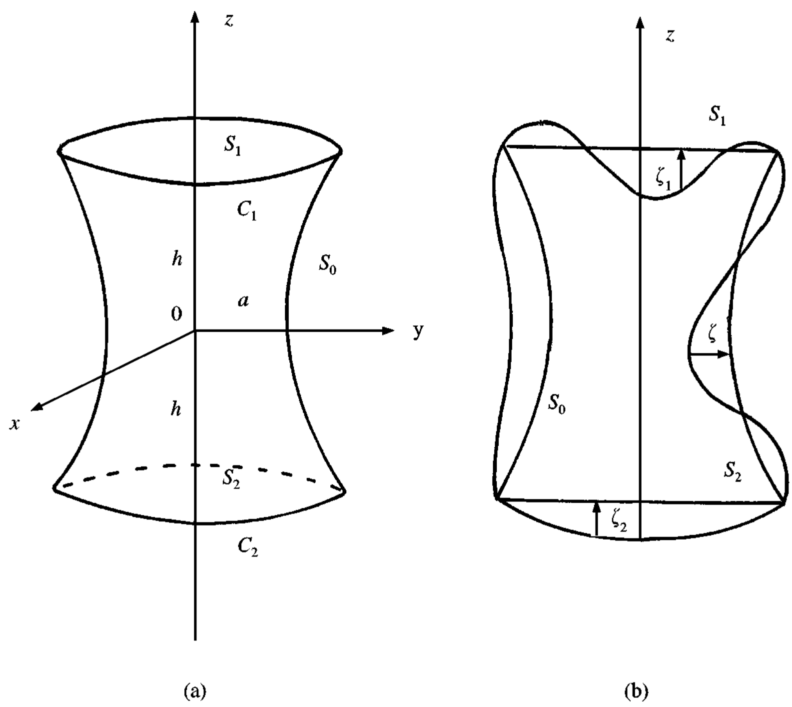


Figure 1. (a) The system under consideration; (b) the deformed system. Definitions of the coordinate system used and some key notations: S_0 free surface; S_1, S_2 membranes with boundaries C_1 and C_2; ζ, ζ_1 are the displcements of the free surface and the membranes.

We use an orthogonal coordinate system Oxyz, Oz being the axis of revolution of the catenoid. The equations of the planes of C_1 and C_2 are, respectively, $z = h$ and $z = -h$; a is the neck radius of the catenoid (Figure 1).

We introduce, instead of the cylindrical coordinates r, θ, z, the coordinates $r, s = a\theta, z$. The equation of S_0 is $r = a\,\mathrm{ch}(z/a)$, the constant volume of the liquid is $\pi a^2 [h + \frac{1}{2}a\,\mathrm{sh}(2h/a)]$ and the element of the surface is $\mathrm{d}S_0 = \mathrm{ch}^2(z/a)\,\mathrm{d}z\,\mathrm{d}s$, and sh and ch are abbreviations for sinh and cosh.

The equations of the perturbed free surface S and of the membranes are

$$r = a\,\mathrm{ch}(z/a) + \zeta(z, s, t) \qquad z = h + \zeta_1(r, s, t), \qquad z = -h + \zeta_2(r, s, t), \tag{1}$$

where ζ, ζ_1, ζ_2 are periodic with respect to s with period $2\pi a$; they as well as their derivatives are considered to be small.

These functions must obviously satisfy the boundary conditions

$$\zeta(\pm h, s, t) = 0; \quad \zeta_1(a, \mathrm{ch}(h/a), s, t) = 0; \quad \zeta_2(a\,\mathrm{ch}(h/a), \mathrm{s}, \mathrm{t}) = 0. \tag{2}$$

We must and the condition which expresses that the volume of the liquid is constant:

$$\int_{S_0} \frac{\zeta}{\mathrm{ch}(z/a)}\,\mathrm{d}S_0 + \int_{S_1} \zeta_1\,\mathrm{d}S_1 - \int_{S_2} \zeta_2\,\mathrm{d}S_2 = 0, \tag{3}$$

the normal displacement of the free surface being $\zeta/\mathrm{ch}(z/a)$.

3. EQUATIONS OF THE PROBLEM

Assuming that the liquid is inviscid and incompressible and its motion is irrotational, we denote by $\phi(r, s, z, t)$ the velocity potential. We have

$$\Delta\phi = 0 \qquad \text{in } \tau, \tag{4}$$

where Δ is the Laplace operator and τ the volume occupied by the liquid in its equilibrium position. ϕ must also satisfy the kinematic conditions

$$\frac{\partial\phi}{\partial n} = \frac{\dot{\zeta}}{\mathrm{ch}(z/a)} \quad \text{on } S_0, \tag{5}$$

$$\frac{\partial\phi}{\partial n} = \dot{\zeta}_1 \quad \text{on } S_1, \tag{6}$$

$$\frac{\partial\phi}{\partial n} = -\dot{\zeta}_2 \quad \text{on } S_2, \tag{7}$$

where $\partial(\,)/\partial n$ is the external normal derivative and the dot indicates derivative with respect to time.

Let us denote by p_0 the constant external pressure and by p the pressure in the liquid. We obtain the first two dynamic conditions by writing the equations of motion of the membranes. If ρ_1 (resp. ρ_2) (in kg/m^2) and T_1 (resp. T_2) (in N/m) are the constant density and the constant tension of S_1 (resp. S_2), we have

$$\rho_1\ddot{\zeta}_1 = T_1\,\Delta\zeta_1 + p - p_0, \tag{8}$$

$$\rho_2\ddot{\zeta}_2 = T_2\,\Delta\zeta_2 + p_0 - p, \tag{9}$$

where the pressure p (in N/m^2) is calculated on S_1 and S_2, respectively. The dynamic condition on the free surface is given by the Laplace law (Landau & Lifschitz 1963; pp. 230–232);

$$p - p_0 = -\alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{10}$$

where α (in N/m) is the surface tension, which we suppose constant, and R_1 and R_2 are the principal radii of curvature of the perturbed free surface, regarded as negative when the centre of curvature lies on the same side of the surface as the fluid.

It is easy to calculate the mean curvature of S, keeping only first-order terms with respect to ζ and its derivatives, by using the general formula (Blaschke 1930); we obtain

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{a\,\mathrm{ch}^2(z/a)}\left\{\frac{\zeta}{a\,\mathrm{ch}(z/a)} + \mathrm{ch}\left(\frac{z}{a}\right)\frac{\partial}{\partial z}\left[\frac{a\zeta_z}{\mathrm{ch}^2(z/a)}\right] + \frac{a\zeta_{ss}}{\mathrm{ch}(z/a)}\right\},$$

where $\zeta_z = (\partial\zeta/\mathrm{d}z)$, $\zeta_{ss} = (\partial^2\zeta/\partial s^2)$.

Using the linearized Bernoulli formula for the pressure

$$p = -\rho(\partial\phi/\partial t) + C(t),$$

we can replace equations (8)–(10) by

$$\rho\left.\frac{\partial\phi}{\partial t}\right|_{S_0} - \frac{\alpha}{a\,\mathrm{ch}^2(z/a)}\left\{\frac{\zeta}{a\,\mathrm{ch}(z/a)} + \mathrm{ch}\left(\frac{z}{a}\right)\frac{\partial}{\partial z}\left[\frac{a\zeta_z}{\mathrm{ch}^2(z/a)}\right] + \frac{a\zeta_{ss}}{\mathrm{ch}(z/a)}\right\} = C(t) - p_0, \tag{11}$$

$$\rho_1\ddot{\zeta}_1 = T_1\,\Delta\zeta_2 - \rho\left.\frac{\partial\phi}{\partial t}\right|_{z=h} - p_2 + C(t), \tag{12}$$

$$\rho_2\ddot{\zeta}_2 = T_2\,\Delta\zeta_2 + \rho\left.\frac{\partial\phi}{\partial t}\right|_{z=-h} + p_0 - C(t), \tag{13}$$

where ρ (in kg/m^3) is the density of the liquid and $C(t)$ an arbitrary function of time. To these, we must obviously add conditions (2) and (3).

In the following *we introduce the space L^2 and the Sobolev space H^1 and H^1_0*, the definition and the properties of which can be found in Kopachevskii *et al.* (1989; pp. 8 and 26–32) and Sanchez & Sanchez (1989; pp. 30, 31).

4. VARIATIONAL EQUATION OF THE PROBLEM

We introduce the auxiliary Neumann problem

$$\Delta\phi = 0 \quad \text{in } \tau; \qquad \frac{\partial\phi}{\partial n} = \frac{g_0}{\mathrm{ch}(z/a)} \quad \text{on } S_0;$$

$$\frac{\partial\phi}{\partial n} = g_1 \quad \text{on } S_1; \qquad \frac{\partial\phi}{\partial n} = g_2 \quad \text{on } S_2;$$

with the compatibility condition

$$\int_{S_0}\frac{g_0}{\mathrm{ch}(z/a)}\,\mathrm{d}S_0 + \int_{S_1} g_1\,\mathrm{d}S_1 + \int_{S_2} g_2\,\mathrm{d}S_2 = 0. \tag{14}$$

We denote by $\mathscr{H}$ the subspace of $\mathscr{L} = L^2(S_0) \times L^2(S_1) \times L^2(S_2)$, the elements $g = \{g_0, g_1, g_2\}$ of which satisfy condition (14) and we equip $\mathscr{H}$ with the scalar product

$$(f, g)_{\mathscr{H}} = \int_{S_0} \frac{f_0 g_0}{\operatorname{ch}(z/a)} \, \mathrm{d}S_0 + \int_{S_1} f_1 g_1 \, \mathrm{d}S_1 + \int_{S_2} f_2 g_2 \, \mathrm{d}S_2;$$

the associated norm $\|\cdot\|_{\mathscr{H}}$ is obviously equivalent to the classical norm of $\mathscr{L}$.

It is well known that, for any $g \in \mathscr{H}$, it is possible to find one and only one function ϕ:

$$\phi \in \tilde{H}^1(\tau) = \left\{ \phi \in H^1(\tau); \int_{S_0} \frac{\phi|_{S_0}}{\operatorname{ch}(z/a)} \, \mathrm{d}S_0 + \int_{S_1} \phi|_{S_1} \, \mathrm{d}S_1 + \int_{S_2} \phi|_{S_2} \, \mathrm{d}S_2 = 0 \right\},$$

a weak solution of the Neumann problem, which satisfies

$$\int_\tau \Delta\phi \, \Delta\psi \, \mathrm{d}\tau = \int_{S_0} \frac{g_0 \psi|_{S_0}}{\operatorname{ch}(z/a)} \, \mathrm{d}S_0 + \int_{S_1} g_1 \psi|_{S_1} \, \mathrm{d}S_1 + \int_{S_2} g_2 \psi|_{S_2} \, \mathrm{d}S_2, \qquad \forall \psi \in \tilde{H}^1(\tau). \quad (15)$$

The traces $\phi|_{S_0}$, $\phi|_{S_1}$, $\phi|_{S_2}$ of ϕ on S_0, S_1, S_2 belong to $L^2(S_0)$, $L^2(S_1)$, $L^2(S_2)$, respectively, and verify condition (14). Consequently, we can introduce a linear operator K of $\mathscr{H}$ into $\mathscr{H}$ defined by

$$\{\phi|_{S_0}, \phi|_{S_1}, \phi|_{S_2}\} = K\{g_0, g_1, g_2\}. \quad (16)$$

It is well known that this operator is self-adjoint, positive definite and compact (Friedmann & Shinbrot 1972).

We write equation (16) in the form

$$\begin{Bmatrix} \phi|_{S_0} \\ \phi|_{S_1} \\ \phi|_{S_2} \end{Bmatrix} = \begin{bmatrix} K_{00} & K_{01} & K_{02} \\ K_{10} & K_{11} & K_{12} \\ K_{20} & K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} g_0 \\ g_1 \\ g_2 \end{Bmatrix},$$

where K_{ij} are operators from $L^2(S_i)$ into $L^2(S_j)$ $(i, j = 1, 2, 3)$, so that

$$\begin{aligned}(Kf, g)_{\mathscr{H}} = &\int_{S_0} (K_{00} f_0 + K_{01} f_1 + K_{02} f_2) \frac{g_0}{\operatorname{ch}(z/a)} \, \mathrm{d}S_0 \\ &+ \int_{S_1} (K_{10} f_0 + K_{11} f_1 + K_{12} f_2) g_1 \, \mathrm{d}S_1 \\ &+ \int_{S_2} (K_{20} f_0 + K_{21} f_1 + K_{22} f_2) \, g_2 \, \mathrm{d}S_2.\end{aligned}$$

Since by virtue of equations (4)–(7), the velocity potential ϕ satisfies the Neumann problem with $g_0 = \dot{\zeta}$, $g_1 = \dot{\zeta}_1$, $g_2 = -\dot{\zeta}_2$, we can write equations (11), (12) and (13) in the form

$$\begin{gathered}\rho(K_{00}\ddot{\zeta} + K_{01}\ddot{\zeta}_1 - K_{02}\ddot{\zeta}_2) \\ - \frac{\alpha}{\operatorname{ch}^2(z/a)} \left[\frac{\zeta}{a \operatorname{ch}(z/a)} + \operatorname{ch}\left(\frac{z}{a}\right) \frac{\partial}{\partial z} \left(\frac{a\zeta_z}{\operatorname{ch}(z/a)} \right) + \frac{a\zeta_{ss}}{\operatorname{ch}(z/a)} \right\} = C(t) - p_0,\end{gathered}$$

$$\rho(K_{10}\ddot{\zeta} + K_{11}\ddot{\zeta}_1 - K_{12}\ddot{\zeta}_2) + \rho_1 \ddot{\zeta}_1 - T_1 \, \Delta\zeta_1 = C(t) - p_0,$$

$$\rho(K_{20}\ddot{\zeta} + K_{21}\ddot{\zeta}_1 - K_{22}\ddot{\zeta}_2) - \rho_2 \ddot{\zeta}_2 + T_2 \, \Delta\zeta_2 = C(t) - p_0.$$

Multiplying these equations, respectively, by $\tilde{\zeta}/\mathrm{ch}(z/a)$, $\tilde{\zeta}_1$, $-\tilde{\zeta}_2$, where $\tilde{\zeta}, \tilde{\zeta}_1, -\tilde{\zeta}_2$ satisfy condition (2) and (14), integrating on S_0, S_1, S_2, adding, taking into account condition (14) and setting

$$\xi = (\zeta, \zeta_1, -\zeta_2), \quad \tilde{\xi} = (\tilde{\zeta}, \tilde{\zeta}_1, -\tilde{\zeta}_2),$$

we obtain the equation

$$\rho(K\ddot{\xi}, \tilde{\xi})_{\mathscr{H}} + \rho_1 \int_{S_1} \ddot{\zeta}_1 \tilde{\zeta}_1 \, \mathrm{d}S_1 + \rho_2 \int_{S_2} \ddot{\zeta}_2 \tilde{\zeta}_2 \, \mathrm{d}S_2 - T_1 \int_{S_1} \Delta\zeta_1 \, \tilde{\zeta}_1 \, \mathrm{d}S_1 - T_2 \int_{S_2} \Delta\zeta_2 \, \tilde{\zeta}_2 \, \mathrm{d}S_2$$
$$= -\alpha \int_{S_0} \frac{1}{a\,\mathrm{ch}^2(z/a)} \left[\frac{\zeta}{a\,\mathrm{ch}(z/a)} + \mathrm{ch}\left(\frac{z}{a}\right) \frac{\partial}{\partial z} \left[\frac{a\zeta_z}{\mathrm{ch}^2(z/a)} \right] + \frac{a\zeta_{ss}}{\mathrm{ch}(z/a)} \right\} \tilde{\zeta} \, \mathrm{d}S_0.$$

The Green formula gives, taking into account equation (2),

$$\int_{S_i} \Delta\zeta_i \, \tilde{\zeta}_i \, \mathrm{d}S_i = \int_{S_i} \nabla\zeta_i \cdot \nabla\tilde{\zeta}_i \, \mathrm{d}S_i, \quad i = 1, 2,$$

where Δ and ∇ are here the classical symbols $\partial^2/\partial x^2 + (\partial^2/\partial y^2)$ and $\mathbf{x}\partial/\partial x + \mathbf{y}(\partial/\partial y)$, $\mathbf{x}$ and $\mathbf{y}$ being the unit vectors of the axes Ox, Oy.

Introducing, instead of S_0 the domain Ω defined by $0 < s < 2\pi_a$, $-h < z < h$, and integrating by parts, we obtain easily, using condition (2) and the periodicity of ζ and $\tilde{\zeta}$, for the coefficient of α the bilinear form

$$m(\zeta, \tilde{\zeta}) = \int_\Omega \frac{\zeta_s\tilde{\zeta}_s + \zeta_z\tilde{\zeta}_z}{\mathrm{ch}^2(z/a)} \, \mathrm{d}z \, \mathrm{d}s - \frac{1}{a^2} \int_\Omega \frac{\zeta\tilde{\zeta}}{\mathrm{ch}^2(z/a)} \, \mathrm{d}z \, \mathrm{d}s. \tag{17}$$

In the following, we denote by $M(\xi, \tilde{\xi})$ the bilinear form defined by

$$\alpha M(\xi, \tilde{\xi}) = \alpha \mathrm{m}(\zeta, \tilde{\zeta}) + T_1 \int_{S_1} \nabla\zeta_1 \cdot \nabla\tilde{\zeta}_1 \, \mathrm{d}S_1 + T_2 \int_{S_2} \nabla\zeta_2 \cdot \nabla\tilde{\zeta}_2 \, \mathrm{d}S_2. \tag{18}$$

We set

$$\mathscr{V} = H^1(S_0) \times H^1(S_1) \times H^2(S_2), \quad V_0 = H_0^1(S_0) \times H_0^1(S_1) \times H_0^1(S_2)$$

and introduce the space

$$V = \left\{ g = \{g_0, g_1, g_2\} \in V_0; \int_{S_0} \frac{g_0}{\mathrm{ch}(z/a)} \, \mathrm{d}S_0 + \int_{S_1} g_1 \, \mathrm{d}S_1 + \int_{S_2} g_2 \, \mathrm{d}S_2 = 0 \right\},$$

equipped with the classical scalar product of $\mathscr{V}$, denote by $(.,.)_{\mathscr{V}}$; H = completion of V for the norm associated to the scalar product

$$(f, g)_{\mathscr{H}} = \rho(Kf, g)_{\mathscr{H}} + \rho_1 \int_{S_1} f_1 g_1 \, \mathrm{d}S_1 + \rho_2 \int_{S_2} f_2 g_2 \, \mathrm{d}S_2.$$

Then, we obtain *the variational formulation* of the problem: To find $\xi(t) \in V$ such that

$$(\ddot{\xi}, \tilde{\xi})_H + \alpha M(\xi, \tilde{\xi}) = 0, \quad \forall \tilde{\xi} \in V. \tag{19}$$

It is easy to prove that $\alpha M(\xi, \xi)$ is the potential energy of the system. Indeed, at first, the $T_i \int_S (\nabla\zeta_i)^2 \, \mathrm{d}S_i$ $(i = 1, 2)$ are the potential energies of the membranes; on the other hand, the potential energy Π of the surface tension forces is given by the formula (Moiseyev & Rumiantsev 1968)

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \alpha \int_{S_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \dot{\zeta} \, \mathrm{d}S_0.$$

Using the expression of the mean curvature of S and performing the integrations by parts which have led to equation (17), we obtain easily

$$\Pi = \alpha m(\zeta, \zeta),$$

and the assertion is proved.

Then, by Rumiantsev theorem (Moiseyev & Rumiantsev 1968), the equilibrium position of the system is stable if the bilinear form $M(.,.)$ is coercive on V, i.e. if there exists a positive constant γ such that $M(\xi, \xi) \geqq \gamma \| \xi \|^2_{\mathscr{V}}$ $\forall \xi \in V$ (Panagiotopoulos 1985).

Consequently, it is necessary to study the coerciveness of the bilinear form $M(.,.)$.

5. STUDY OF THE BILINEAR FORM $M(.,.)$

In order to study the coerciveness of this form, we seek

$$\inf_{\xi \in V} \frac{\displaystyle\int_{\Omega} \frac{\zeta_s^2 + \zeta_z^2}{\mathrm{ch}^2(z/z)} \mathrm{d}z \, \mathrm{d}s + \frac{T_1}{\alpha} \int_{S_1} (\nabla \zeta_1)^2 \, \mathrm{d}S_1 + \frac{T_2}{\alpha} \int_{S_2} (\nabla \zeta_2)^2 \, \mathrm{d}S_2}{\displaystyle\int_{\Omega} \frac{\zeta^2}{a^2 \, \mathrm{ch}^2(z/a)} \mathrm{d}z \, \mathrm{d}s}.$$

This greater lower bound λ exists and is positive or zero. We are going to study it by a method which can be found in Roseau (1987); see also Riesz & Nagy (1968; pp. 244–248).

It is well known that there exists a sequence $\{\xi_n\} = \{\zeta_n, \zeta_{1n}, -\zeta_{2n}\} \in V$, such that

$$\lambda = \lim_{n \to \infty} \frac{\displaystyle\int_{\Omega} \frac{\zeta_{ns}^2 + \zeta_{nz}^2}{\mathrm{ch}^2(z/a)} \mathrm{d}z \, \mathrm{d}s + \frac{T_1}{\alpha} \int_{S_1} (\nabla \zeta_{1n})^2 \, \mathrm{d}S_1 + \frac{T_2}{\alpha} \int_{S_2} (\nabla \zeta_{2n})^2 \, \mathrm{d}S_2}{\displaystyle\int_{\Omega} \frac{\zeta_n^2}{a^2 \, \mathrm{ch}^2(z/a)} \mathrm{d}z \, \mathrm{d}s}.$$

By homogeneity, we may suppose the denominator to be equal to one. Then, since the square roots of the terms of the numerator are norms in $H_0^1(S_0)$, $H_0^1(S_1)$, $H_0^1(S_2)$, the sequence $\{\xi_n\}$ is bounded in V_0 and it is possible to find a subsequence, denoted again by $\{\xi_n\}$, which converges weakly in V_0 and strongly in $\mathscr{L}$ to a limit $\Sigma = \{\sigma, \sigma_1, -\sigma_2\}$. It is easy to see that $\int_{\Omega} \sigma^2 / a^2 \, \mathrm{ch}^2(z/a) \, \mathrm{d}z \, \mathrm{d}s = 1$ and the Σ satisfies condition (14).

Let us prove that

$$\lambda = \frac{\displaystyle\int_{\Omega} \frac{\sigma_s^2 + \sigma_z^2}{\mathrm{ch}^2(z/a)} \mathrm{d}z \, \mathrm{d}s + \frac{T_1}{\alpha} \int_{S_1} (\nabla \sigma_1)^2 \, \mathrm{d}S_1 + \frac{T_2}{\alpha} \int_{S_2} (\nabla \sigma_2)^2 \, \mathrm{d}S_2}{\displaystyle\int_{\Omega} \frac{\sigma^2}{a^2 \, \mathrm{ch}^2(z/a)} \mathrm{d}z \, \mathrm{d}s}.$$

By the definition of λ, the ratio is greater than or equal to λ. We are going to prove that it is smaller than λ. We have easily that

$$\begin{aligned}
&\int_\Omega \frac{\zeta_{ns}^2+\zeta_{nz}^2}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s+\frac{T_1}{\alpha}\int_{S_1}(\nabla\zeta_{1n})^2\,\mathrm{d}S_1+\frac{T_2}{\alpha}\int_{S_2}(\nabla\zeta_{2n})^2\,\mathrm{d}S_2\\
&\quad\geqslant\int_\Omega \frac{\sigma_s^2+\sigma_z^2}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s+\frac{T_1}{\alpha}\int_{S_1}(\nabla\sigma_1)^2\,\mathrm{d}S_1+\frac{T_2}{\alpha}\int_{S_2}(\nabla\sigma_2)^2\,\mathrm{d}S_2\\
&\quad\quad+2\left\{\int_\Omega \frac{(\zeta_{ns}-\sigma_s)\sigma_s+(\zeta_{nz}-\sigma_z)\sigma_z}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s+\frac{T_1}{\alpha}\int_{S_1}\nabla(\zeta_{1n}-\sigma_1)\cdot\nabla\sigma_1\,\mathrm{d}S_1\right.\\
&\quad\quad\left.+\frac{T_2}{\alpha}\int_{S_1}\nabla(\zeta_{2n}-\sigma_2)\cdot\nabla\sigma_2\,\mathrm{d}S_2\right\}. \qquad (20)
\end{aligned}$$

The bilinear form

$$\hat{M}(U,\tilde{U})=\int_\Omega \frac{u_s\tilde{u}_s+v_s\tilde{v}_s}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s+\frac{T_1}{\alpha}\int_{S_1}\nabla u_1\cdot\nabla\tilde{u}_1\,\mathrm{d}S_1+\frac{T_2}{\alpha}\int_{S_2}\nabla u_2\cdot\nabla\tilde{u}_2\,\mathrm{d}S_2,$$

with $U=\{u,u_1,-u_2\}$, $\tilde{U}=\{\tilde{u},\tilde{u}_1,-\tilde{u}_2\}$ is bounded in $\mathscr{V}$; then, there exists an operator $\mathscr{A}$ from $\mathscr{V}$ into $\mathscr{V}$ such that

$$\hat{M}(U,\tilde{U})=(U,\mathscr{A}\tilde{U})_{\mathscr{V}}.$$

Consequently, the term between braces in equation (20) is $(\xi_n-\Sigma,\mathscr{A}\Sigma)_{\mathscr{V}}$, and it converges to zero, since the sequence $\{\xi_n\}$ converges weakly to Σ in $\mathscr{V}$. The assertion is proved by taking $n\to\infty$ in inequality (20).

It is noted that λ is different from zero. Indeed, if $\lambda=0$, Σ is constant and this constant is zero by virtue of condition (14), and it is in contradiction with $\int_\Omega \sigma^2/a^2\,\mathrm{ch}^2(z/a)\,\mathrm{d}z\,\mathrm{d}s=1$.

By the definition of λ, we have the inequality

$$\int_\Omega \frac{\zeta_s^2+\zeta_z^2}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s+\frac{T_1}{\alpha}\int_{S_1}(\nabla\zeta_1)^2\,\mathrm{d}S_1+\frac{T_2}{\alpha}\int_{S_2}(\nabla\zeta_2)^2\,\mathrm{d}S_2-\lambda\int_\Omega\frac{\zeta^2}{a^2\,\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s\geqslant 0$$

for any $\xi\in V$. Setting $\xi=\Sigma+\varepsilon\,\delta\xi$, $\varepsilon\in\mathbb{R}$, and carrying out in the preceding inequality which must be verified for any $\varepsilon\in\mathbb{R}$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{\zeta_s\,\delta\zeta_s+\zeta_z\,\delta\zeta_z}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s+\frac{T_1}{\alpha}\int_{S_1}\nabla\zeta_1\cdot\nabla(\delta\zeta_1)\,\mathrm{d}S_1\\
&\quad+\frac{T_2}{\alpha}\int_{S_2}\nabla\zeta_2\cdot\nabla(\delta\zeta_2)\,\mathrm{d}S_2-\lambda\int_\Omega\frac{\zeta\,\delta\zeta}{a^2\,\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s=0
\end{aligned}$$

for any $\delta\xi\in V$. Introducing a multiplier μ associated in condition (14), we replace this equation by

$$\begin{aligned}
&\int_\Omega \frac{\zeta_s\,\delta\zeta_s+\zeta_z\,\delta\zeta_z}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s+\frac{T_1}{\alpha}\int_{S_1}\nabla\zeta_1\cdot\nabla(\delta\zeta_1)\,\mathrm{d}S_1+\frac{T_2}{\alpha}\int_{S_2}\nabla\zeta_2\cdot\nabla(\delta\zeta_2)\,\mathrm{d}S_2\\
&\quad-\lambda\int_\Omega\frac{\zeta\,\delta\zeta}{a^2\,\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s-\mu\left[\int_\Omega \mathrm{ch}(z/a)\,\delta\zeta\,\mathrm{d}z\,\mathrm{d}s+\int_{S_1}\delta\zeta_1\,\mathrm{d}S_1-\int_{S_2}\delta\zeta_2\,\mathrm{d}S_2\right]=0
\end{aligned}$$

for any $\delta\xi \in V_0$. Taking $\zeta \in \mathscr{D}(S_0)$, $\zeta_1 \in \mathscr{D}(S_1)$, $\zeta_2 \in \mathscr{D}(S_2)$ we obtain

$$\frac{\zeta_{ss}}{\mathrm{ch}^2(z/a)} + \frac{\zeta}{\partial z}\left[\frac{\zeta_z}{\mathrm{ch}^2(z/a)}\right] + \lambda\,\frac{\zeta}{a^2\,\mathrm{ch}^2(z/a)} + \mu\,\mathrm{ch}(z/a) = 0 \quad \text{on } S_0, \tag{21}$$

$$\Delta\zeta_1 = -\frac{\mu\alpha}{T_1} \quad \text{on } S_1, \tag{22}$$

$$\Delta\zeta_2 = \frac{\mu\alpha}{T_2} \quad \text{on } S_2, \tag{23}$$

in the distributed sense. However, by virtue of Schwartz's theorem for elliptic equations, we have $\zeta, \zeta_1, \zeta_2 \in \mathscr{C}^\infty$.

We must obviously add to this conditions (2) and the periodicity conditions.

The solutions of equations (22) and (23) with $\zeta_1 = 0$ on C_1 and $\zeta_2 = 0$ on C_2 are

$$\zeta_1 = -\frac{\mu\alpha}{4T_1}\left[r^2 - a^2\,\mathrm{ch}^2\frac{h}{a}\right], \qquad \zeta_2 = \frac{\mu\alpha}{4T_2}\left[r^2 - a^2\,\mathrm{ch}^2\frac{h}{a}\right];$$

Condition (14) gives

$$\int_\Omega \zeta\,\mathrm{ch}\left(\frac{z}{a}\right)\mathrm{d}z\,\mathrm{d}s = \int_{S_2}\zeta_2\,\mathrm{d}S_2 - \int_{S_1}\zeta_1\,\mathrm{d}S_1 = -\mu\,\frac{\pi\alpha a^4\,\mathrm{ch}^4(h/a)}{8}\left(\frac{1}{T_1} + \frac{1}{T_2}\right).$$

Setting

$$K_0 = \frac{8}{\pi\alpha a^4\,\mathrm{ch}^4\left(\frac{h}{a}\right)\left(\frac{1}{T_1} + \frac{1}{T_2}\right)} > 0,$$

we have

$$\mu = -K_0\int_\Omega \zeta\,\mathrm{ch}(z/a)\,\mathrm{d}z\,\mathrm{d}s.$$

Finally, the problem is reduced to following eigenvalue problem:

$$\left.\begin{aligned}\frac{\zeta_{ss}}{\mathrm{ch}^2(z/a)} + \frac{\partial}{\partial z}\left[\frac{\zeta_2}{\mathrm{ch}^2(z/a)}\right] + \lambda\,\frac{\zeta}{a^2\,\mathrm{ch}^2(z/a)} - K_0\,\mathrm{ch}\left(\frac{z}{a}\right)\int_\Omega \zeta\,\mathrm{ch}\left(\frac{z}{a}\right)\mathrm{d}z\,\mathrm{d}s = 0,\\ \zeta(\pm h, s) = 0;\end{aligned}\right\} \tag{24}$$

ζ is the classical solution of the problem, and it is periodic with respect to s, with period $2\pi a$.

In order to solve problem (24), we seek solutions in the form

$$\zeta = S(s)Z(z)$$

and hence obtain

$$\frac{S_{ss}Z}{\mathrm{ch}^2(z/a)} + S\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{Z_z}{\mathrm{ch}^2\,(z/a)} + \frac{\lambda SZ}{a^2\,\mathrm{ch}^2(z/a)} - K_0\,\mathrm{ch}\left(\frac{z}{a}\right)\int_0^{2\pi a} S(s)\,\mathrm{d}s\int_{-h}^{h} Z(z)\,\mathrm{ch}\left(\frac{z}{a}\right)\mathrm{d}z = 0,\right.$$

$$S(s) \text{ with the period } 2\pi a, \qquad Z(\pm h) = 0.$$

Integrating with respect to s between 0 and $2\pi a$ and taking into account the periodicity, we obtain

$$\int_0^{2\pi a} S(s)\,\mathrm{d}s \left\{\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{Z_s}{\mathrm{ch}^2\,(z/a)}\right] + \frac{\lambda Z}{a^2\,\mathrm{ch}^2(z/a)} - 2\pi a K_0\,\mathrm{ch}\left(\frac{z}{a}\right)\int_{-h}^{h} Z(z)\,\mathrm{ch}\left(\frac{z}{a}\right)\mathrm{d}z\right\} = 0.$$

We must distinguish between the following two cases.

Case (a) for which

$$\int_0^{2\pi a} S(s)\,\mathrm{d}s = 0.$$

In this case, we have

$$-\frac{S_{ss}}{S} = \frac{\dfrac{\mathrm{d}}{\mathrm{d}z}\left[\dfrac{Z_z}{\mathrm{ch}^2(z/a)}\right] + \dfrac{\lambda Z}{a^2\,\mathrm{ch}^2(z/a)}}{\dfrac{Z}{\mathrm{ch}^2(z/a)}} = \text{constant}.$$

We obtain

$$S_n(s) = A_n \cos\left(\frac{n}{a}\,s\right) + B_n \sin\left(\frac{n}{a}\,s\right); \quad A_n,\ B_n \text{ constant};$$

and

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{Z_z}{\mathrm{ch}^2(z/a)}\right] + \frac{\lambda - n^2}{a^2\,\mathrm{ch}^2(z/a)}\ z = 0,\\ &Z(\pm h) = 0;\ n = 1, 2, 3, \ldots \end{aligned}\right\} \tag{25}$$

Problem (25) for $Z(z)$ is a classical Strum–Liouville problem (Courant & Hilbert 1965). For each value of $n = 1, 2, \ldots,$ we have a sequence of eigen-values, $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nm}, \ldots$, strictly greater than n^2, and, hence strictly greater than one. On the other hand, for each n, the eigenfunctions Z_{nm} form an orthogonal basis in the space $\hat{L}^2(-h, h)$, the space of the functions $Z \in L^2(-h, h)$ equipped with the scalar product

$$(Z, \hat{Z})_{\tilde{L}^2} = \int_{-h}^{h} \frac{Z\hat{Z}}{a^2\,\mathrm{ch}^2(z/a)}\,\mathrm{d}z.$$

Case (b) for which

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{Z_z}{\mathrm{ch}^2(z/a)}\right] + \frac{\lambda Z}{a^2\,\mathrm{ch}^2(z/a)} - 2\pi a K_0\,\mathrm{ch}\left(\frac{z}{a}\right)\int_{-h}^{h} Z(z)\,\mathrm{ch}\left(\frac{z}{a}\right)\mathrm{d}z = 0.$$

In this case, we can write the equation in the form

$$\frac{\dfrac{\mathrm{d}^2}{\mathrm{d}s^2}\left[S(s) - \dfrac{1}{2\pi a}\displaystyle\int_0^{2\pi a} S(s)\,\mathrm{d}s\right]}{S(s) - \dfrac{1}{2\pi a}\displaystyle\int_0^{2\pi a} S(s)\,\mathrm{d}s} = 2\pi a K_0\,\frac{\mathrm{ch}^3\left(\dfrac{z}{a}\right)}{Z}\int_{-h}^{h} Z(z)\,\mathrm{ch}\left(\frac{z}{a}\right)\mathrm{d}z = \text{constant}.$$

It is possible only for $S(s) - (1/2\pi a)\int_0^{2\pi a} S(s)\,\mathrm{d}s = 0$, and then, $S(s) = \text{constant}$. Setting

$$\tilde{\mu} = -\,2\pi a K_0 \int_{-h}^{h} Z(z)\,\mathrm{ch}\left(\frac{z}{a}\right)\mathrm{d}z, \tag{26}$$

We obtain the following problem for $Z(z)$:

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{Z_z}{\mathrm{ch}^2(z/a)}\right] + \frac{\lambda Z}{a^2\,\mathrm{ch}^2(z/a)} + \tilde{\mu}\,\mathrm{ch}\left(\frac{z}{a}\right) = 0, \\ Z(\pm h) = 0. \end{aligned}\right\} \tag{27}$$

Multiplying equation (27) by $\hat{Z} \in H_0^1(-h, h)$, and integrating in the interval $(-h, h)$ we obtain after an integration by parts

$$a(Z, \hat{Z}) = \lambda(Z, \hat{Z})_{\hat{L}^2}$$

with

$$a(Z, \hat{Z}) = \int_{-h}^{h} \frac{Z_z \hat{Z}_z}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z + 2\pi a^2 K_0 \int_{-h}^{h} Z\,\mathrm{ch}\left(\frac{z}{a}\right)\mathrm{d}z \int_{-h}^{h} Z\,\mathrm{ch}\left(\frac{z}{a}\right)\mathrm{d}z.$$

The embedding $H_0^1(-h, h) \in \hat{L}^2(-h, h)$ is obviously continuous, dense and compact; the bilinear form $a(Z, \hat{Z})$ is symmetrical, continuous and coercive in $H_0^1(-h, h)$ by virtue of Poincaré's inequality (Sanchez & Sanchez 1989); see also Velte (1976; pp. 59–64). Consequently, equation (27) is a classical eigenvalue problem and its eigenfunctions form an orthogonal basis in $\hat{L}^2(-h, h)$.

Considering problems (25) and (27), remarking that the functions 1, $\cos(ns/a)$, $\sin(ns/a)$, $(n = 1, 2, \dots)$ form an orthogonal basis in $L^2(0, 2\pi a)$ and using a classical theorem (Courant & Hilbert 1965; Vol. 1, pp. 56, 57), we see that by method of separation of variables we obtain all the eigenvalues and all the eigenfunctions of problem (24).

Let us now solve problem (27). Setting

$$x = \frac{z}{a}, \quad Y = \frac{z}{\mathrm{ch}\,x}, \quad \beta = \frac{h}{a}, \quad \lambda = 1 - \mathrm{th}^2\,\omega',$$

we have the equivalent problem

$$\left.\begin{aligned} y_{xx} - (2\,\mathrm{th}^2\,x - 2 + \mathrm{th}^2\,\omega')y = -\,\tilde{\tilde{\mu}}\,\mathrm{ch}^2\,x, \quad \tilde{\tilde{\mu}} = a^2\tilde{\mu}, \\ y(\pm\beta) = 0, \end{aligned}\right\} \tag{28}$$

the general solution of which we can obtain explicitly. Indeed, the homogeneous equation is a degenerate Lamé equation, and $y = -\,[\tilde{\tilde{\mu}}/(4 - \mathrm{th}^2\,\omega')]\,\mathrm{ch}^2\,x$ is a particular solution of the complete equation.

We must distinguish the cases (i) ω' real and (ii) $\omega' = \mathrm{i}\Omega'$ with Ω' real.

For ω' real, excluding $\omega' = +\infty$ which gives $\lambda = 0$, we must consider two cases:

Subcase 1: $\omega' = 0$, *then* $\lambda = 1$. The general solution of equation (28) is

$$y = A\,\mathrm{th}\,x \quad (x - \coth x) + B\,\mathrm{th}\,x - \tfrac{1}{4}\,\tilde{\tilde{\mu}}\,\mathrm{ch}^2\,x; \quad A,\ B \text{ constants};$$

the conditions $y(\pm\beta)=0$ give $B=0$, $\tilde{\tilde{\mu}}=4A[\mathrm{sh}\,\beta\,(\beta\text{-}\coth\beta)/\mathrm{ch}^3\,\beta]$, and consequently

$$Z(z)=A\left[\mathrm{sh}\,\frac{z}{a}\left(\frac{z}{a}-\coth\frac{z}{a}\right)-\frac{\mathrm{sh}\,\beta\;(\beta-\coth\beta)}{\mathrm{ch}^3\,\beta}\,\mathrm{ch}^3\,\frac{z}{a}\right].$$

Substituting this expression into equation (26), we obtain after a few calculations the equation

$$A\left\{4\,\mathrm{th}\,\beta\,(\beta-\coth\beta)+\frac{\pi a^4K_0}{2}\left[-3\beta^2\,\mathrm{th}\,\beta+\beta(1-\mathrm{sh}^2\,\beta)(1-2\,\mathrm{sh}^2\,\beta)\right.\right.$$
$$\left.\left.-\,\mathrm{sh}\,\beta\,\mathrm{ch}\,\beta\,(1+4\,\mathrm{sh}^2\,\beta)\right]\right\}=0; \qquad (29)$$

$\lambda=1$ cannot be eigenvalue if the quantity between braces is not zero. The coefficient of $\frac{1}{2}\pi a^4K_0$ has been studied by Erle *et al.* (1970); its first positive root is $\beta=2{\cdot}23918$ and it is negative for $0<\beta<2{\cdot}23918$. On the other hand, $\beta-\coth\beta$ is positive (resp. negative) if $\beta>1{\cdot}9997$ (resp. $<1{\cdot}9997$). Consequently, taking $0<\beta<2{\cdot}23918$, $\lambda=1$ is impossible if $0<\beta<1{\cdot}9997$; or $1{\cdot}9997<\beta<2{\cdot}23918$, and

$$\frac{\pi a^4K_0}{2}\neq\frac{4\,\mathrm{th}\,\beta(\beta-\coth\beta)}{3\beta^2\,\mathrm{th}\,\beta-\beta(1-\mathrm{sh}^2\,\beta)(1-2\,\mathrm{sh}^2\,\beta)+\mathrm{sh}\,\beta\,\mathrm{ch}\,\beta(1+4\,\mathrm{sh}^2\,\beta)}.$$

Subcase 2: $\omega'\neq 0,\ \neq\infty$; *then* $0<\lambda<1$. Now, the general solution of equation (28) is

$$y=A(\mathrm{th}\,\omega'\,\mathrm{ch}\,x+\mathrm{sh}\,x)\mathrm{e}^{-x\,\mathrm{th}\,\omega'}+B(\mathrm{th}\,\omega'\,\mathrm{ch}\,x-\mathrm{sh}\,x)\mathrm{e}^{x\,\mathrm{th}\,\omega'}-\frac{\tilde{\tilde{\mu}}}{4-\mathrm{th}^2\,\omega'}\,\mathrm{ch}^3\,x.$$

At first, the conditions $y(\pm\beta)=0$ give

$$(A-\mathrm{B})\left[(\mathrm{th}\,\omega'\,\mathrm{ch}\,\beta+\mathrm{sh}\,\beta)\mathrm{e}^{-\beta\,\mathrm{th}\,\omega'}-(\mathrm{th}\,\omega'\,\mathrm{ch}\,\beta-\mathrm{sh}\,\beta)\mathrm{e}^{\beta\,\mathrm{th}\,\omega'}\right]=0.$$

The coefficient of $A-B$ is different from zero, because, setting $\mathrm{th}\,\omega'=u$, $0<u<1$, it is easy to see graphically that the equation $\mathrm{e}^{2\beta u}=(u+\mathrm{th}\,\beta)/(u-\mathrm{th}\,\beta)$ has no roots. Consequently, $A=B$ and, as in subcase 1, we find

$$Z(z)=A\left\{\left(\mathrm{th}\,\omega'\,\mathrm{ch}\frac{z}{a}+\mathrm{sh}\frac{z}{a}\right)\mathrm{e}^{-(z/a)\,\mathrm{th}\,\omega'}+\left(\mathrm{th}\,\omega'\,\mathrm{ch}\,\frac{z}{a}-\mathrm{sh}\,\frac{z}{a}\right)\mathrm{e}^{(z/a)\,\mathrm{th}\,\omega'}\right.$$
$$\left.-\frac{(\mathrm{th}\,\omega'\,\mathrm{ch}\,\beta+\mathrm{sh}\,\beta)\mathrm{e}^{-\beta\,\mathrm{th}\,\omega'}+(\mathrm{th}\,\omega'\,\mathrm{ch}\,\beta-\mathrm{sh}\,\beta)\mathrm{e}^{\beta\,\mathrm{th}\,\omega'}}{\mathrm{ch}^3\,\beta}\,\mathrm{ch}^3\,\frac{z}{a}\right\}.$$

Substituting into equation (24), we obtain an equation of the form

$$A\,[F(\omega',\beta)+2\pi a^4K_0\tilde{F}(\omega',\beta]=0.$$

There is no eigenvalue λ between 0 and 1 if the coefficient of A is not zero. After some long calculations and not obvious transformations, the equation $F+2\pi a^4K_0\tilde{F}=0$ can be written, setting $\mathrm{th}\,\omega'=u$, $0<u<1$, as

$$G_\beta(u)=H_\beta(u), \qquad (30)$$

with

$$H_\beta(u) = \frac{\frac{9}{2}[2 + \pi a^4 K_0(\frac{1}{4}\,\mathrm{sh}\,4\beta - \beta)]}{u^2 - 1},$$

$$G_\beta(u) = \frac{4\pi a^4 K_0\,\mathrm{ch}^4\beta}{u/\mathrm{th}(\beta u)\ \ \mathrm{th}\,\beta} - (u^2 - 1) - [\tfrac{1}{2}\pi a^4 K_0(3\beta + 2\,\mathrm{sh}\,2\beta + \tfrac{1}{4}\,\mathrm{sh}\,4\beta) - 6].$$

The function $H_\beta(u)$ is strictly decreasing from $-\frac{9}{2}[2 + \pi a^4 K_0(\frac{1}{4}\,\mathrm{sh}\,4\beta - \beta)] < 0$ to $-\infty$.
It is easy to see that the function $F_\beta(u) = u/\mathrm{th}(\beta u) - \mathrm{th}\,\beta$ is strictly increasing, strictly positive if $0 < \beta < 1{\cdot}9997$, and taking the value zero at some point u_0 if $\beta > 1{\cdot}9997$. Then $G_\beta(u)$ is strictly decreasing; besides, we have $G_\beta(1 \geq \frac{3}{4}\pi a^4 K_0(\mathrm{sh}\,2\beta - 2\beta) + 6 \geq 6$. Using the graphs of functions $G_\beta(u)$ and $H_\beta(u)$, we easily obtain the following results:

if $0 < \beta < 1{\cdot}9997$, $G_\beta(u)$ is positive and equation (20) has no root;
If $1{\cdot}9997 < \beta < 2{\cdot}23918$, the equation has no root if $G_\beta(0) < H_\beta(0)$, i.e. if

$$\frac{\pi a^4 K_0}{2} > \frac{4\,\mathrm{th}\,\beta(\beta - \coth\beta)}{3\beta^2\,\mathrm{th}\,\beta - \beta(1 - \mathrm{sh}^2\beta)(1 - 2\,\mathrm{sh}^2\beta) + \mathrm{sh}\,\beta\,\mathrm{ch}\,\beta(1 + 4\,\mathrm{sh}^2\beta)}.$$

Finally, for ω' real, there is no eigenvalue λ between 0 and 1 if

$$0 < \beta < 2{\cdot}23918, \tag{31}$$

as obtained by Strube (1992) and Capodanno (1995) for rigid disks; setting

$$\frac{2}{T} = \frac{1}{T_1} + \frac{1}{T_2}$$

and then

$$K_0 = \frac{4T}{\pi\alpha a^4\,\mathrm{ch}^4\beta},$$

this gives

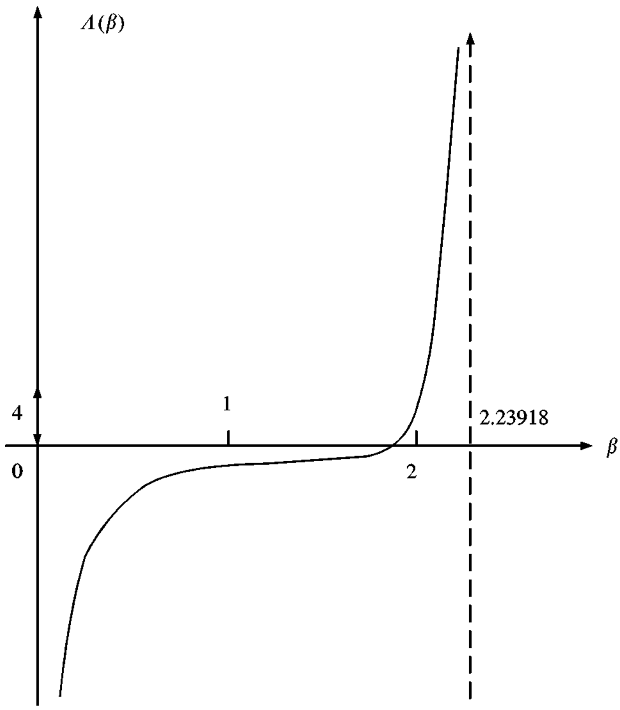
$$\frac{T}{\alpha} > \frac{2\,\mathrm{th}\,\beta\,\mathrm{ch}^4\beta(\beta - \coth\beta)}{3\beta^2\,\mathrm{th}\,\beta - \beta(1 - \mathrm{sh}^2\beta)(1 - 2\,\mathrm{sh}^2\beta) + \mathrm{sh}\,\beta\,\mathrm{ch}\,\beta(1 + 4\,\mathrm{sh}^2\beta)}, \tag{32}$$

which is a new result.

Inequality (32) is obviously verified if $0 < \beta < 1{\cdot}9997$. If $1{\cdot}9997 < \beta < 2{\cdot}23918$, it expresses that the ratio between the harmonic mean of the tensions of the membranes and the surface tension must be sufficiently high. We have plotted the curve which represents the right-hand side $\Lambda(\beta)$ of equation (32) for $0 < \beta < 2{\cdot}23928$ and calculated a few values of this function, as shown in Figure 2.

Let us consider now the case $\omega' = \mathrm{i}\Omega'$, with Ω' real, obviously different from zero. We have $\lambda = 1 + \mathrm{tg}^2\,\Omega'$ and the corresponding eigenvalues λ are strictly greater than one, where tg is an abbreviation for tangent (tan). We are going to show that we can determine these eigenvalues graphically. As in the last case that we have studied, replacing $\mathrm{th}\,\omega'$ by $\mathrm{i}\,\mathrm{tg}\,\Omega'$, we obtain

$$(A - B)[(\mathrm{i}\,\mathrm{tg}\,\Omega' + \mathrm{th}\,\beta)\mathrm{e}^{-\mathrm{i}\beta\,\mathrm{tg}\,\Omega'} - (\mathrm{i}\,\mathrm{tg}\,\Omega' - \mathrm{th}\,\beta)\mathrm{e}^{i\beta\,\mathrm{tg}\,\Omega'}] = 0.$$



β	Λ (B)
0·5	−1·77077
1	−0·20106
2	2·90901
2·1	5·62546
2·2	24·32184

Figure 2. The curve $\Lambda(\beta)$ as a function of β, for $0 < \beta < 2{\cdot}23928$, with table giving some of the values.

We have a first group of eigenvalues by putting the bracketed expression equal to zero. Setting $\operatorname{tg} \Omega' = u > 0$, we find the equation

$$\operatorname{tg}(\beta u) = -\operatorname{th} \beta/u,$$

which has a denumerable infinity of roots u_n: $[(2n-1)]/2\beta < u_n < (n\pi/\beta)$, $n = 1, 2, \ldots$. Then, we obtain the eigenvalues $\lambda = 1 + u_n^2$.

The case $A = B$ leads to an equation analogous to equation (30); setting again $\operatorname{tg} \Omega' = u > 0$, we obtain

$$G_\beta^0(u) = H_\beta^0(u), \tag{33}$$

with

$$H_\beta^0(u) = -\frac{\frac{9}{2}[2 + \pi a^4 K_0(\frac{1}{4}\operatorname{sh} 4\beta - \beta)]}{u^2 + 1},$$

$$G_\beta^0(u) = \frac{4\pi a^4 K_0 \operatorname{ch}^4 \beta}{u/\operatorname{tg}(\beta u) - \operatorname{th} \beta} + (u^2 + 1) - \left[\frac{\pi a^4 K_0}{2}(3\beta + 2\operatorname{sh} 2\beta + \tfrac{1}{4}\operatorname{sh} 4\beta) - 6\right];$$

$H_\beta^0(u)$ is negative and strictly increasing from $-\frac{9}{2}[2 + \pi a^2 K_0(\frac{1}{4}\operatorname{sh} 4\beta - \beta)] < 0$ to zero. $G_\beta^0(u)$ is strictly increasing and is infinite for $u = u_n'$, $(n\pi)/\beta < u_n' < [(2n+1)]/2\beta$ $(n = 1, 2, \ldots)$, roots of the equation $\operatorname{tg}(\beta u) = u/\operatorname{th} \beta$.

On the other hand, the difference

$$G^0_\beta(0) - H^0_\beta(0) = \frac{16(1-\beta\,\mathrm{th}\,\beta) + 2\pi a^4 K_0[3\beta^2\,\mathrm{tg}\,\beta - \beta(1-\mathrm{sh}^2\,\beta)(1-2\mathrm{sh}^2\,\beta) + \mathrm{sh}\,\beta\,\mathrm{ch}\,\beta(1+4\mathrm{sh}^2\,\beta)]}{1-\beta\,\mathrm{th}\,\beta}$$

is positive if $0 < \beta < 1{\cdot}9997$ and negative if $1{\cdot}9997 < \beta < 2{\cdot}23918$ by virtue of inequality (32).

Thus, equation (33) has a denumerable infinity of roots u''_n, $u'_n < u''_n < u'_{n+1}$ ($n = 1, 2, \ldots$ if $0 < \beta < 1{\cdot}9997$ and $n = 0, 1, 2, \ldots$, $u'_0 = 0$ if $1{\cdot}9997 < \beta < 2{\cdot}23918$), and the corresponding eigenvalues are $\lambda = 1 + u''^2_n$.

Finally, under conditions (31) *and* (32), *the smallest eigenvalue of problem* (24) *is strictly greater than one.*

By the definition of λ, we can write

$$\int_\Omega \frac{\zeta_s^2 + \zeta_z^2}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s + \frac{T_1}{\alpha}\int_{S_1}(\nabla\zeta_1)^2\,\mathrm{d}S_1 + \frac{T_2}{\gamma}\int_{S_2}(\nabla\zeta_2)^2\,\mathrm{d}S_2 \geq \lambda\int_\Omega \frac{\zeta^2}{a^2\,\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s \quad \forall\xi\in V,$$

with $\lambda > 1$. With $0 < \varepsilon < 1$, we have

$$\begin{aligned} M(\xi,\xi) = {} & \varepsilon\left[\int_\Omega \frac{\zeta_s^2 + \zeta_z^2}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s + \frac{T_1}{\alpha}\int_{S_1}(\nabla\zeta_1)^2\,\mathrm{d}S_1 + \frac{T_2}{\alpha}\int_{S_2}(\nabla\zeta_2)^2\,\mathrm{d}S_2\right] \\ & + (1-\varepsilon)\left[\int_\Omega \frac{\zeta_s^2 + \zeta_z^2}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s + \frac{T_1}{\alpha}\int_{S_1}(\nabla\zeta_1)^2\,\mathrm{d}S_1 + \frac{T_2}{\alpha}\int_{S_2}(\nabla\zeta_2)^2\,\mathrm{d}S_2\right] \\ & - \int_\Omega \frac{\zeta^2}{a^2\,\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s. \end{aligned}$$

Using the preceding inequality, we have

$$\begin{aligned} M(\xi,\xi) \geq {} & \varepsilon\left[\int_\Omega \frac{\zeta_s^2 + \zeta_z^2}{\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s + \frac{T_1}{\alpha}\int_{S_1}(\nabla\zeta_1)^2\,\mathrm{d}S_1 + \frac{T_2}{\alpha}\int_{S_2}(\nabla\zeta_2)^2\,\mathrm{d}S_2\right] \\ & + [(1-\varepsilon)\,\lambda - 1]\int_\Omega \frac{\zeta^2}{a^2\,\mathrm{ch}^2(z/a)}\,\mathrm{d}z\,\mathrm{d}s. \end{aligned}$$

Since $\lambda > 1$, we can choose ε such that the quantity between brackets is positive: $0 < \varepsilon < (\lambda - 1)/\lambda$.

Therefore, it is possible to find a positive constant γ so that

$$M(\xi,\xi) \geq \gamma\,\|\xi\|^2, \quad \forall\xi\in V.$$

Finally, under conditions (31) *and* (32), *the bilinear form* $M(.,.)$ *is coercive in* V.

6. EXISTENCE OF THE EIGENFREQUENCIES OF THE SYSTEM

Let us consider again the variational equation of problem (19), i.e.

$$(\ddot{\xi}, \tilde{\xi})_H + \alpha M(\xi, \tilde{\xi}) = 0, \quad \forall\tilde{\xi}\in V.$$

The bilinear form ($M(.,.)$ is, obviously, symmetrical and continuous in V; under the preceding two conditions, it is coercive in V.

On the other hand, it is easy to prove that the embedding $V \subset H$, which is dense by construction, is continuous and compact. It is continuous, because the embeddings $H_0^1(S_i) \subset L^2(S_i)$ ($i = 0, 1, 2$) are continuous and K is a continuous operator in $\mathscr{H}$.

Let us consider now a sequence $\{g^n\} \in V$, which converges weakly for $n \to \infty$ to $g \in V$, and therefore, by the Rellich theorem, strongly to g in the space $\mathscr{L}$; we have

$$\| g^n - g \|_H^2 \leq \rho \| K \| \; \| g^n - g \|_{\mathscr{H}}^2 + \rho_1 \| g_1^n - g_1 \|_{L^2(S_1)}^2 + \rho_2 \| g_2^n - g_2 \|_{L^2(S_2)}^2.$$

so that g^n converges strongly to g in H. Consequently, the embedding $V \subset H$ is compact.

Therefore, the problem is a standard vibration problem (Sanchez & Sanchez 1989), i.e. there exists a denumerable infinity of positive eigenvalues

$$0 < \omega_1^2 \leq \omega_2^2 \leq \cdots \leq \pm \omega_n^2 \leq \cdots, \quad \omega_n^2 \to \infty,$$

and the associated eigenfunctions $\xi_1, \xi_2, \ldots \xi_n, \ldots$ form an orthogonal basis in $\mathscr{L}$ and in V, equipped with the scalar product $M(.,.)$.

We can add the following remark. For the numerical calculation of the eigenvalues by means of the Rayleigh ratio

$$\frac{\alpha M(\xi, \xi)}{(\xi, \xi)_H} = \frac{\alpha M(\xi, \xi)}{\rho (K\xi, \xi)_H + \rho_1 \| \zeta_1 \|_{L^2(S_1)}^2 + \rho_2 \| \zeta_2 \|_{L^2(S_2)}^2},$$

there is only one difficulty, because it is impossible, in general, to determine explicitly the operator K. But we known that, for each $\xi \in V$, there exists $\phi \in \tilde{H}^1(\tau)$, a weak solution of a Neumann problem, such that $(K\xi, \xi)_{\mathscr{H}} = \int_\tau (\nabla \phi)^2 \, d\tau$, and it is well known that ϕ can be calculated approximately, for instance by the Rayleigh–Ritz method.

7. CONCLUSION

Under the simple conditions (31) and (32), we have prove the stability of the system of liquid membranes and the existence of its eigenfrequencies. The first condition expresses that the ratio between the distance of the centres of the disks and the neck radius of the catenoid be sufficiently small, exactly smaller than 4·47836. The second condition is identically satisfied if this ratio lies between zero and 3·994. If the ratio lies between 3·9994 and 4·47836, this condition expresses that the ratio between the harmonic mean of the tensions of the membranes and the surface tension must be sufficiently great.

We can obtain analogous results in the simpler case of the cylindrical bridge. It should be interesting to study the cases of onduloidal and nodoidal bridges; however, these problems will certainly lead to complicated calculations because of the presence of elliptic functions.

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